Chapter 4

Deflection and Stiffness

Faculty of Engineering
Mechanical Dept.
# Chapter Outline

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**Spring Rate**

- *Elasticity* – property of a material that enables it to regain its original configuration after deformation
- *Spring* – a mechanical element that exerts a force when deformed

---

**Fig. 4–1**
Spring Rate

- Relation between force and deflection, \( F = F(y) \)
- *Spring rate*

\[
k(y) = \lim_{{\Delta y \to 0}} \frac{\Delta F}{\Delta y} = \frac{dF}{dy} \tag{4-1}
\]

- For linear springs, \( k \) is constant, called *spring constant*

\[
k = \frac{F}{y} \tag{4-2}
\]
Tension, Compression, and Torsion

- Total extension or contraction of a uniform bar in tension or compression

\[ \delta = \frac{Fl}{AE} \]  \hspace{1cm} (4–3)

- Spring constant, with \( k = \frac{F}{\delta} \)

\[ k = \frac{AE}{l} \]  \hspace{1cm} (4–4)
Angular deflection (in radians) of a uniform solid or hollow round bar subjected to a twisting moment $T$

$$\theta = \frac{Tl}{GJ} \quad (4-5)$$

Converting to degrees, and including $J = \pi d^4/32$ for round solid

$$\theta = \frac{583.6Tl}{Gd^4} \quad (4-6)$$

Torsional spring constant for round bar

$$k = \frac{T}{\theta} = \frac{GJ}{l} \quad (4-7)$$
Deflection Due to Bending

• Curvature of beam subjected to bending moment $M$
  \[ \frac{1}{\rho} = \frac{M}{EI} \quad \text{where } \rho \text{ is the radius of curvature.} \quad \text{(4-8)} \]

• From mathematics, curvature of plane curve
  \[ \frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \quad \text{(4-9)} \]

• Slope of beam at any point $x$ along the length
  \[ \theta = \frac{dy}{dx} \]

• If the slope is very small, the denominator of Eq. (4-9) approaches unity.

• Combining Eqs. (4-8) and (4-9), for beams with small slopes,
  \[ \frac{M}{EI} = \frac{d^2y}{dx^2} \]
Deflection Due to Bending

- Recall Eqs. (3-3) and (3-4)

\[ V = \frac{dM}{dx} \quad (3-3) \]

\[ \frac{dV}{dx} = \frac{d^2M}{dx^2} = q \quad (3-4) \]

- Successively differentiating

\[ \frac{M}{EI} = \frac{d^2y}{dx^2} \]

\[ \frac{V}{EI} = \frac{d^3y}{dx^3} \]

\[ \frac{q}{EI} = \frac{d^4y}{dx^4} \]
Deflection Due to Bending

\[
\frac{q}{EI} = \frac{d^4 y}{dx^4} \quad (4-10)
\]

\[
\frac{V}{EI} = \frac{d^3 y}{dx^3} \quad (4-11)
\]

\[
\frac{M}{EI} = \frac{d^2 y}{dx^2} \quad (4-12)
\]

\[
\theta = \frac{dy}{dx} \quad (4-13)
\]

\[
y = f(x) \quad (4-14)
\]
Example 4-1

For the beam in Fig. 4–2, the bending moment equation, for $0 \leq x \leq l$, is

$$M = \frac{wl}{2}x - \frac{w}{2}x^2$$

Using Eq. (4–12), determine the equations for the slope and deflection of the beam, the slopes at the ends, and the maximum deflection.

Fig. 4–2
Example 4-1

Integrating Eq. (4–12) as an indefinite integral we have

\[ EI \frac{dy}{dx} = \int M \, dx = \frac{wl}{4} x^2 - \frac{w}{6} x^3 + C_1 \]  \hspace{1cm} (1)

where \( C_1 \) is a constant of integration that is evaluated from geometric boundary conditions. We could impose that the slope is zero at the midspan of the beam, since the beam and loading are symmetric relative to the midspan. However, we will use the given boundary conditions of the problem and verify that the slope is zero at the midspan. Integrating Eq. (1) gives

\[ EIy = \int \int M \, dx = \frac{wl}{12} x^3 - \frac{w}{24} x^4 + C_1 x + C_2 \]  \hspace{1cm} (2)

The boundary conditions for the simply supported beam are \( y = 0 \) at \( x = 0 \) and \( l \). Applying the first condition, \( y = 0 \) at \( x = 0 \), to Eq. (2) results in \( C_2 = 0 \). Applying the second condition to Eq. (2) with \( C_2 = 0 \),

\[ EIy(l) = \frac{wl}{12} l^3 - \frac{w}{24} l^4 + C_1 l = 0 \]
Example 4-1

Solving for $C_1$ yields $C_1 = -wl^3/24$. Substituting the constants back into Eqs. (1) and (2) and solving for the deflection and slope results in

$$y = \frac{wx}{24EI} (2lx^2 - x^3 - l^3)$$

(3)

$$\theta = \frac{dy}{dx} = \frac{w}{24EI} (6lx^2 - 4x^3 - l^3)$$

(4)

Comparing Eq. (3) with that given in Table A–9, beam 7, we see complete agreement. For the slope at the left end, substituting $x = 0$ into Eq. (4) yields

$$\theta|_{x=0} = -\frac{wl^3}{24EI}$$

and at $x = l$,

$$\theta|_{x=l} = \frac{wl^3}{24EI}$$

At the midspan, substituting $x = l/2$ gives $dy/dx = 0$, as earlier suspected.
Example 4-1

The maximum deflection occurs where $dy/dx = 0$. Substituting $x = l/2$ into Eq. (3) yields

$$y_{\text{max}} = -\frac{5wl^4}{384EI}$$

which again agrees with Table A–9–7.

7 Simple supports—uniform load

$$R_1 = R_2 = \frac{wl}{2} \quad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\text{max}} = -\frac{5wl^4}{384EI}$$
Beam Deflection Methods

• Some of the more common methods for solving the integration problem for beam deflection
  ◦ Superposition
  ◦ Moment-area method
  ◦ Singularity functions
  ◦ Numerical integration

• Other methods that use alternate approaches
  ◦ Castigliano energy method
  ◦ Finite element software
Beam Deflection by Superposition

- **Superposition** determines the effects of each load separately, then adds the results.

- Separate parts are solved using any method for simple load cases.

- Many load cases and boundary conditions are solved and available in Table A-9, or in references such as *Roark’s Formulas for Stress and Strain*.

- **Conditions**
  - Each effect is linearly related to the load that produces it.
  - A load does not create a condition that affects the result of another load.
  - The deformations resulting from any specific load are not large enough to appreciably alter the geometric relations of the parts of the structural system.
Table A-9
Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force $V$ and bending moment $M$ follow the sign conventions given in Sec. 3-2.)

1 Cantilever—end load

$R_1 = V = F$ \hspace{1cm} $M_1 = Fl$

$M = F(x - l)$

$y = \frac{Fx^2}{6EI} (x - 3l)$

$y_{max} = -\frac{Fl^3}{3EI}$

2 Cantilever—intermediate load

$R_1 = V = F$ \hspace{1cm} $M_1 = Fa$

$M_{AB} = F(x - a)$ \hspace{1cm} $M_{BC} = 0$

$y_{AB} = \frac{Fx^2}{6EI} (x - 3a)$

$y_{BC} = \frac{Fa^2}{6EI} (a - 3x)$

$y_{max} = \frac{Fa^2}{6EI} (a - 3l)$
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Shear, Moment, and Deflection of Beams (Continued)</td>
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<tr>
<td>(Note: Force and moment reactions are positive in the directions shown; equations for shear force $V$ and bending moment $M$ follow the sign conventions given in Sec. 3–2.)</td>
</tr>
</tbody>
</table>

### 3 Cantilever—uniform load

- **Equations:**
  - $R_1 = wl$
  - $M_1 = \frac{wl^2}{2}$
  - $V = w(l - x)$
  - $M = -\frac{w}{2}(l - x)^2$
  - $y = \frac{wx^2}{24EI} (4lx - x^2 - 6l^2)$
  - $y_{max} = -\frac{wl^4}{8EI}$

### 4 Cantilever—moment load

- **Equations:**
  - $R_1 = V = 0$
  - $M_1 = M = M_B$
  - $y = \frac{M_B x^2}{2EI}$
  - $y_{max} = \frac{M_B l^2}{2EI}$
Table A-9
Shear, Moment, and Deflection of Beams
(Continued)
(Note: Force and moment reactions are positive in the directions shown; equations for shear force \( V \) and bending moment \( M \) follow the sign conventions given in Sec. 3–2.)

5 Simple supports—center load

\[
R_1 = R_2 = \frac{F}{2}
\]

\[
V_{AB} = R_1 \quad V_{BC} = -R_2
\]

\[
M_{AB} = \frac{Fx}{2} \quad M_{BC} = \frac{F}{2}(l - x)
\]

\[
y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)
\]

\[
y_{\text{max}} = -\frac{Fl^3}{48EI}
\]

6 Simple supports—intermediate load

\[
R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}
\]

\[
V_{AB} = R_1 \quad V_{BC} = -R_2
\]

\[
M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l - x)
\]

\[
y_{AB} = \frac{Fbx}{6EI}(x^2 + b^2 - l^2)
\]

\[
y_{BC} = \frac{Fa(l - x)}{6EI}(x^2 + a^2 - 2lx)
\]
### Table A-9

Shear, Moment, and Deflection of Beams

*(Continued)*

*(Note: Force and moment reactions are positive in the directions shown; equations for shear force $V$ and bending moment $M$ follow the sign conventions given in Sec. 3–2.)*

#### 7 Simple supports—uniform load

- **Equations:**
  - $R_1 = R_2 = \frac{w l}{2}$
  - $V = \frac{w l}{2} - w x$
  - $M = \frac{w x}{2} (l - x)$
  - $y = \frac{w x}{24 E I} (2l x^2 - x^3 - l^3)$
  - $y_{\text{max}} = -\frac{5 w l^4}{384 E I}$

#### 8 Simple supports—moment load

- **Equations:**
  - $R_1 = R_2 = \frac{M_B}{l}$
  - $V = \frac{M_B}{l}$
  - $M_{AB} = \frac{M_B x}{l}$
  - $M_{BC} = \frac{M_B}{l} (x - l)$
  - $y_{AB} = \frac{M_B x}{6 E I l} (x^2 + 3a^2 - 6al + 2l^2)$
  - $y_{BC} = \frac{M_B}{6 E I l} [x^3 - 3l x^2 + x (2l^2 + 3a^2) - 3a^2 l]$
9 Simple supports—twin loads

\[ R_1 = R_2 = F \quad V_{AB} = F \quad V_{BC} = 0 \]
\[ V_{CD} = -F \]
\[ M_{AB} = Fx \quad M_{BC} = Fa \quad M_{CD} = F(l - x) \]

\[ y_{AB} = \frac{Fx}{6EI} (x^2 + 3a^2 - 3la) \]
\[ y_{BC} = \frac{Fa}{6EI} (3x^2 + a^2 - 3lx) \]
\[ y_{max} = \frac{Fa}{24EI} (4a^2 - 3l^2) \]

10 Simple supports—overhanging load

\[ R_1 = \frac{Fa}{l} \quad R_2 = \frac{F}{l} (l + a) \]
\[ V_{AB} = -\frac{Fa}{l} \quad V_{BC} = F \]
\[ M_{AB} = -\frac{F(ax)}{l} \quad M_{BC} = F(x - l - a) \]

\[ y_{AB} = \frac{F(ax)}{6EI} (l^2 - x^2) \]
\[ y_{BC} = \frac{F(x - l)}{6EI} [(x - l)^2 - a(3x - l)] \]
\[ y_c = -\frac{Fa^2}{3EI} (l + a) \]
Table A-9
Shear, Moment, and Deflection of Beams
(Continued)
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3–2.)

11 One fixed and one simple support—center load

\[ R_1 = \frac{11F}{16} \quad R_2 = \frac{5F}{16} \quad M_1 = \frac{3Fl}{16} \]

\[ V_{AB} = R_1 \quad V_{BC} = -R_2 \]

\[ M_{AB} = \frac{F}{16} (11x - 3l) \quad M_{BC} = \frac{5F}{16} (l - x) \]

\[ y_{AB} = \frac{Fx^2}{96EI} (11x - 9l) \]

\[ y_{BC} = \frac{F(l - x)}{96EI} (3x^2 + 2l^2 - 10lx) \]

12 One fixed and one simple support—intermediate load

\[ R_1 = \frac{Fb}{2l} (3l^2 - b^2) \quad R_2 = \frac{Fa^2}{2l^3} (3l - a) \]

\[ M_1 = \frac{Fb}{2l^2} (l^2 - b^2) \]

\[ M_{AB} = \frac{Fb}{2l^2} \left[b^2 l - l^3 + x(3l^2 - b^2)\right] \]

\[ M_{BC} = \frac{Fa^2}{2l^3} (3l^2 - 3lx - al + ax) \]

\[ y_{AB} = \frac{Fbx^2}{12EI} \left[3l(b^2 - l^2) + x(3l^2 - b^2)\right] \]

\[ y_{BC} = y_{AB} - \frac{F(x - a)^3}{6EI} \]
Table A-9
Shear, Moment, and Deflection of Beams
(Continued)
(Note: Force and moment reactions are positive in the directions shown; equations for shear force \( V \) and bending moment \( M \) follow the sign conventions given in Sec. 3–2.)

13 One fixed and one simple support—uniform load

\[
R_1 = \frac{5wl}{8} \quad R_2 = \frac{3wl}{8} \quad M_1 = \frac{wl^2}{8}
\]

\[
V = \frac{5wl}{8} - wx
\]

\[
M = -\frac{w}{8}(4x^2 - 5lx + l^2)
\]

\[
y = \frac{wx^2}{48EI}(l-x)(2x-3l)
\]

14 Fixed supports—center load

\[
R_1 = R_2 = \frac{F}{2} \quad M_1 = M_2 = \frac{Fl}{8}
\]

\[
V_{AB} = -V_{BC} = \frac{F}{2}
\]

\[
M_{AB} = \frac{F}{8}(4x-l) \quad M_{BC} = \frac{F}{8}(3l-4x)
\]

\[
y_{AB} = \frac{Fx^2}{48EI}(4x-3l)
\]

\[
y_{max} = -\frac{Fl^3}{192EI}
\]
Table A-9
Shear, Moment, and Deflection of Beams (Continued)
(Note: Force and moment reactions are positive in the directions shown; equations for shear force $V$ and bending moment $M$ follow the sign conventions given in Sec. 3–2.)

15 Fixed supports—intermediate load

$$R_1 = \frac{Fb^2}{l^3}(3a + b) \quad R_2 = \frac{Fa^2}{l^3}(3b + a)$$

$$M_1 = \frac{F ab^2}{l^2} \quad M_2 = \frac{Fa^2b}{l^2}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb^2}{l^3}[x(3a + b) - al]$$

$$M_{BC} = M_{AB} - F(x - a)$$

$$y_{AB} = \frac{F b x^2}{6EI l^2}[x(3a + b) - 3al]$$

$$y_{BC} = \frac{F a^2(l - x)^2}{6EI l^2}[(l - x)(3b + a) - 3bl]$$

16 Fixed supports—uniform load

$$R_1 = R_2 = \frac{w l}{2} \quad M_1 = M_2 = \frac{w l^2}{12}$$

$$V = \frac{w}{2}(l - 2x)$$

$$M = \frac{w}{12}(6lx - 6x^2 - l^2)$$

$$y = -\frac{wx^2}{24EI}(l - x)^2$$

$$y_{max} = \frac{w l^4}{384EI}$$
Consider the uniformly loaded beam with a concentrated force as shown in Fig. 4–3. Using superposition, determine the reactions and the deflection as a function of $x$. 

Fig. 4–3
Example 4-2

**Solution**
Considering each load state separately, we can superpose beams 6 and 7 of Table A–9.

6 Simple supports—intermediate load

\[
R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}
\]

\[
V_{AB} = R_1 \quad V_{BC} = -R_2
\]

\[
M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l-x)
\]

\[
y_{AB} = \frac{Fbx}{6El}(x^2 + b^2 - l^2)
\]

\[
y_{BC} = \frac{Fa(l-x)}{6El}(x^2 + a^2 - 2lx)
\]

7 Simple supports—uniform load

\[
R_1 = R_2 = \frac{wl}{2} \quad V = \frac{wl}{2} - wx
\]

\[
M = \frac{wx}{2}(l-x)
\]

\[
y = \frac{wx}{24El} \left(2lx^2 - x^3 - l^3\right)
\]

\[
y_{max} = -\frac{5wl^4}{384El}
\]
Example 4-2

For the reactions we find

\[ R_1 = \frac{Fb}{l} + \frac{wl}{2} \]

\[ R_2 = \frac{Fa}{l} + \frac{wl}{2} \]

The loading of beam 6 is discontinuous and separate deflection equations are given for regions \(AB\) and \(BC\). Beam 7 loading is not discontinuous so there is only one equation. Superposition yields

\[ y_{AB} = \frac{Fbx}{6EI} (x^2 + b^2 - l^2) + \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]

\[ y_{BC} = \frac{Fa(l-x)}{6EI} (x^2 + a^2 - 2lx) + \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \]
Example 4-3

Consider the beam in Fig. 4–4a and determine the deflection equations using superposition.

Fig. 4–4
Example 4-3

For region $AB$ we can superpose beams 7 and 10 of Table A–9 to obtain

$$y_{AB} = \frac{wx}{24EI}(2lx^2 - x^3 - l^3) + \frac{F_ax}{6EI}(l^2 - x^2)$$

10 Simple supports—overhanging load

$$R_1 = \frac{Fa}{l} \quad R_2 = \frac{F}{l}(l + a)$$

$$V_{AB} = -\frac{Fa}{l} \quad V_{BC} = F$$

$$M_{AB} = -\frac{F ax}{l} \quad M_{BC} = F(x - l - a)$$

$$y_{AB} = \frac{F ax}{6EI}(l^2 - x^2)$$

$$y_{BC} = \frac{F(x - l)}{6EI}[(x - l)^2 - a(3x - l)]$$

$$y_C = -\frac{Fa^2}{3EI}(l + a)$$

7 Simple supports—uniform load

$$R_1 = R_2 = \frac{wl}{2} \quad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\text{max}} = -\frac{5wl^4}{384EI}$$
Example 4-3

For region $BC$, how do we represent the uniform load? Considering the uniform load only, the beam deflects as shown in Fig. 4–4b. Region $BC$ is straight since there is no bending moment due to $w$. The slope of the beam at $B$ is $\theta_B$ and is obtained by taking the derivative of $y$ given in the table with respect to $x$ and setting $x = l$. Thus,

$$\frac{dy}{dx} = \frac{d}{dx} \left[ \frac{wx}{24EI} (2lx^2 - x^3 - l^3) \right] = \frac{w}{24EI} (6lx^2 - 4x^3 - l^3)$$
Example 4-3

Substituting \( x = l \) gives

\[ \theta_B = \frac{w}{24EI} (6ll^2 - 4l^3 - l^3) = \frac{wl^3}{24EI} \]

The deflection in region \( BC \) due to \( w \) is \( \theta_B (x - l) \), and adding this to the deflection due to \( F \), in \( BC \), yields

\[ y_{BC} = \frac{wl^3}{24EI} (x - l) + \frac{F(x - l)}{6EI} [(x - l)^2 - a(3x - l)] \]
Distributed Load on Beam

- Distributed load $q(x)$ called *load intensity*
- Units of force per unit length
Beam Deflection by Singularity Functions

Relationships between Load, Shear, and Bending

\[ V = \frac{dM}{dx} \]  
\[ \frac{dV}{dx} = \frac{d^2 M}{dx^2} = q \]  
\[ \int_{VA}^{VB} dV = V_B - V_A = \int_{XA}^{XB} q \, dx \]  
\[ \int_{MA}^{MB} dM = M_B - M_A = \int_{XA}^{XB} V \, dx \]

- The change in shear force from \( A \) to \( B \) is equal to the area of the loading diagram between \( x_A \) and \( x_B \).
- The change in moment from \( A \) to \( B \) is equal to the area of the shear-force diagram between \( x_A \) and \( x_B \).
Beam Deflection by Singularity Functions

- A notation useful for integrating across discontinuities
- Angle brackets indicate special function to determine whether forces and moments are active

### Table 3–1

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<th>Function</th>
<th>Graph of $f_n(x)$</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>Concentrated moment</td>
<td>$(x-a)^{-2}$</td>
<td>$\langle x-a \rangle^{-2} = 0 \quad x \neq a$</td>
</tr>
<tr>
<td>(unit doublet)</td>
<td></td>
<td>$\langle x-a \rangle^{-2} = +\infty \quad x = a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\int \langle x-a \rangle^{-2} , dx = \langle x-a \rangle^{-1}$</td>
</tr>
<tr>
<td>Concentrated force</td>
<td>$(x-a)^{-1}$</td>
<td>$\langle x-a \rangle^{-1} = 0 \quad x \neq a$</td>
</tr>
<tr>
<td>(unit impulse)</td>
<td></td>
<td>$\langle x-a \rangle^{-1} = +\infty \quad x = a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\int \langle x-a \rangle^{-1} , dx = \langle x-a \rangle^{0}$</td>
</tr>
<tr>
<td>Unit step</td>
<td>$(x-a)^{0}$</td>
<td>$\langle x-a \rangle^{0} = \begin{cases} 0 &amp; x &lt; a \ 1 &amp; x \geq a \end{cases}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\int \langle x-a \rangle^{0} , dx = \langle x-a \rangle^{1}$</td>
</tr>
<tr>
<td>Ramp</td>
<td>$(x-a)^{1}$</td>
<td>$\langle x-a \rangle^{1} = \begin{cases} 0 &amp; x &lt; a \ x-a &amp; x \geq a \end{cases}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\int \langle x-a \rangle^{1} , dx = \frac{(x-a)^{2}}{2}$</td>
</tr>
</tbody>
</table>


*Shigley’s Mechanical Engineering Design*
Example 4-5

Consider beam 6 of Table A–9, which is a simply supported beam having a concentrated load \( F \) not in the center. Develop the deflection equations using singularity functions.

6 Simple supports—intermediate load

\[
R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}
\]

\[
V_{AB} = R_1 \quad V_{BC} = -R_2
\]

\[
M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l - x)
\]

\[
y_{AB} = \frac{Fbx}{6EIl}(x^2 + b^2 - l^2)
\]

\[
y_{BC} = \frac{Fa(l - x)}{6EIl}(x^2 + a^2 - 2lx)
\]
Example 4-5

First, write the load intensity equation from the free-body diagram,

\[ q = R_1 \langle x \rangle^{-1} - F \langle x - a \rangle^{-1} + R_2 \langle x - l \rangle^{-1} \]  \hspace{1cm} (1)

Integrating Eq. (1) twice results in

\[ V = R_1 \langle x \rangle^{0} - F \langle x - a \rangle^{0} + R_2 \langle x - l \rangle^{0} \]  \hspace{1cm} (2)

\[ M = R_1 \langle x \rangle^{1} - F \langle x - a \rangle^{1} + R_2 \langle x - l \rangle^{1} \]  \hspace{1cm} (3)

Recall that as long as the \( q \) equation is complete, integration constants are unnecessary for \( V \) and \( M \); therefore, they are not included up to this point. From statics, setting \( V = M = 0 \) for \( x \) slightly greater than \( l \) yields \( R_1 = Fb/l \) and \( R_2 = Fa/l \). Thus Eq. (3) becomes

\[ M = \frac{Fb}{l} \langle x \rangle^{1} - F \langle x - a \rangle^{1} + \frac{Fa}{l} \langle x - l \rangle^{1} \]
Example 4-5

Integrating Eqs. (4–12) and (4–13) as indefinite integrals gives

\[ M = \frac{Fb}{l} \langle x \rangle^1 - F \langle x - a \rangle^1 + \frac{Fa}{l} \langle x - l \rangle^1 = EI \frac{d^2 y}{dx^2} \]

\[ EI \frac{dy}{dx} = \frac{Fb}{2l} \langle x \rangle^2 - \frac{F}{2} \langle x - a \rangle^2 + \frac{Fa}{2l} \langle x - l \rangle^2 + C_1 \]

\[ EI y = \frac{Fb}{6l} \langle x \rangle^3 - \frac{F}{6} \langle x - a \rangle^3 + \frac{Fa}{6l} \langle x - l \rangle^3 + C_1 x + C_2 \]

Note that the first singularity term in both equations always exists, so \( \langle x \rangle^2 = x^2 \) and \( \langle x \rangle^3 = x^3 \). Also, the last singularity term in both equations does not exist until \( x = l \), where it is zero, and since there is no beam for \( x > l \) we can drop the last term. Thus

\[ EI \frac{dy}{dx} = \frac{Fb}{2l} x^2 - \frac{F}{2} \langle x - a \rangle^2 + C_1 \] \hspace{1cm} (4)

\[ EI y = \frac{Fb}{6l} x^3 - \frac{F}{6} \langle x - a \rangle^3 + C_1 x + C_2 \] \hspace{1cm} (5)
Example 4-5

The constants of integration $C_1$ and $C_2$ are evaluated by using the two boundary conditions $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. The first condition, substituted into Eq. (5), gives $C_2 = 0$ (recall that $(0 - a)^3 = 0$). The second condition, substituted into Eq. (5), yields

$$0 = \frac{Fb}{6l} l^3 - \frac{F}{6} (l - a)^3 + C_1 l = \frac{Fbl^2}{6} - \frac{Fb^3}{6} + C_1 l$$

Solving for $C_1$ gives

$$C_1 = -\frac{Fb}{6l} (l^2 - b^2)$$

Finally, substituting $C_1$ and $C_2$ in Eq. (5) and simplifying produces

$$y = \frac{F}{6EIl} [bx(x^2 + b^2 - l^2) - l(x - a)^3]$$

(6)

Comparing Eq. (6) with the two deflection equations for beam 6 in Table A–9, we note that the use of singularity functions enables us to express the deflection equation with a single equation.
Example 4-6

Determine the deflection equation for the simply supported beam with the load distribution shown in Fig. 4–6.
Example 4-6

This is a good beam to add to our table for later use with superposition. The load intensity equation for the beam is

\[ q = R_1 \langle x \rangle^{-1} - w \langle x \rangle^0 + w \langle x - a \rangle^0 + R_2 \langle x - l \rangle^{-1} \]  

(1)

where the \( w \langle x - a \rangle^0 \) is necessary to “turn off” the uniform load at \( x = a \).

From statics, the reactions are

\[ R_1 = \frac{wa}{2l} (2l - a) \quad R_2 = \frac{wa^2}{2l} \]  

(2)

For simplicity, we will retain the form of Eq. (1) for integration and substitute the values of the reactions in later.

Two integrations of Eq. (1) reveal

\[ V = R_1 \langle x \rangle^0 - w \langle x \rangle^1 + w \langle x - a \rangle^1 + R_2 \langle x - l \rangle^0 \]  

(3)

\[ M = R_1 \langle x \rangle^1 - \frac{w}{2} \langle x \rangle^2 + \frac{w}{2} \langle x - a \rangle^2 + R_2 \langle x - l \rangle^1 \]  

(4)
As in the previous example, singularity functions of order zero or greater starting at $x = 0$ can be replaced by normal polynomial functions. Also, once the reactions are determined, singularity functions starting at the extreme right end of the beam can be omitted. Thus, Eq. (4) can be rewritten as

$$M = R_1x - \frac{w}{2}x^2 + \frac{w}{2}(x - a)^2$$

(5)

Integrating two more times for slope and deflection gives

$$EI \frac{dy}{dx} = \frac{R_1}{2}x^2 - \frac{w}{6}x^3 + \frac{w}{6}(x - a)^3 + C_1$$

(6)

$$EIy = \frac{R_1}{6}x^3 - \frac{w}{24}x^4 + \frac{w}{24}(x - a)^4 + C_1x + C_2$$

(7)
Example 4-6

The boundary conditions are $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. Substituting the first condition in Eq. (7) shows $C_2 = 0$. For the second condition

\[ 0 = \frac{R_1}{6} l^3 - \frac{w}{24} l^4 + \frac{w}{24} (l - a)^4 + C_1 l \]

Solving for $C_1$ and substituting into Eq. (7) yields

\[ EIy = \frac{R_1}{6} x(x^2 - l^2) - \frac{w}{24} x(x^3 - l^3) - \frac{w}{24l} x(l - a)^4 + \frac{w}{24}(x - a)^4 \]

Finally, substitution of $R_1$ from Eq. (2) and simplifying results gives

\[ y = \frac{w}{24EI} [2ax(2l - a)(x^2 - l^2) - xl(x^3 - l^3) - x(l - a)^4 + l(x - a)^4] \]
Strain Energy

- External work done on elastic member in deforming it is transformed into *strain energy*, or *potential energy*.
- Strain energy equals product of average force and deflection.

\[
U = \frac{F}{2} y = \frac{F^2}{2k}
\]  

(4–15)
Some Common Strain Energy Formulas

- For axial loading, applying \( k = AE/l \) from Eq. (4-4),

\[
U = \frac{F^2 l}{2AE} \quad \text{tension and compression (4-16)}
\]

or

\[
U = \int \frac{F^2}{2AE} \, dx \quad \text{tension and compression (4-17)}
\]

- For torsional loading, applying \( k = GJ/l \) from Eq. (4-7),

\[
U = \frac{T^2 l}{2GJ} \quad \text{torsion (4-18)}
\]

or

\[
U = \int \frac{T^2}{2GJ} \, dx \quad \text{torsion (4-19)}
\]
Some Common Strain Energy Formulas

- For direct shear loading,
  \[ U = \frac{F^2 l}{2AG} \]  \hspace{1cm} \text{(4–20)}
  or
  \[ U = \int \frac{F^2}{2AG} \, dx \]  \hspace{1cm} \text{direct shear (4–21)}

- For bending loading,
  \[ U = \frac{M^2 l}{2EI} \]  \hspace{1cm} \text{(4–22)}
  or
  \[ U = \int \frac{M^2}{2EI} \, dx \]  \hspace{1cm} \text{bending (4–23)}
Some Common Strain Energy Formulas

- For transverse shear loading,

\[ U = \frac{CV^2 l}{2AG} \quad \text{transverse shear} \]

or

\[ U = \int \frac{CV^2}{2AG} \, dx \]  

where \( C \) is a modifier dependent on the cross sectional shape.

Table 4-1

<table>
<thead>
<tr>
<th>Strain-Energy Correction Factors for Transverse Shear</th>
<th>Beam Cross-Sectional Shape</th>
<th>Factor C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Circular</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td>Thin-walled tubular, round</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>Box sections†</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>Structural sections‡</td>
<td>1.00</td>
</tr>
</tbody>
</table>

†Use area of web only.
Summary of Common Strain Energy Formulas

\[ U = \frac{F^2l}{2AE} \]  \{ tension and compression \}

\[ U = \int \frac{F^2}{2AE} \, dx \]

\[ U = \frac{T^2l}{2GJ} \]  \{ torsion \}

\[ U = \int \frac{T^2}{2GJ} \, dx \]

\[ U = \frac{M^2l}{2EI} \]  \{ bending \}

\[ U = \int \frac{M^2}{2EI} \, dx \]

\[ U = \frac{F^2l}{2AG} \]  \{ direct shear \}

\[ U = \int \frac{F^2}{2AG} \, dx \]

\[ U = \frac{CV^2l}{2AG} \]  \{ transverse shear \}

\[ U = \int \frac{CV^2}{2AG} \, dx \]
Example 4-8

A cantilever beam with a round cross section has a concentrated load $F$ at the end, as shown in Fig. 4–9a. Find the strain energy in the beam.

Fig. 4–9
Example 4-8

To determine what forms of strain energy are involved with the deflection of the beam, we break into the beam and draw a free-body diagram to see the forces and moments being carried within the beam. Figure 4–9a shows such a diagram in which the transverse shear is $V = -F$, and the bending moment is $M = -Fx$. The variable $x$ is simply a variable of integration and can be defined to be measured from any convenient point. The same results will be obtained from a free-body diagram of the right-hand portion of the beam with $x$ measured from the wall. Using the free end of the beam usually results in reduced effort since the ground reaction forces do not need to be determined.

Fig. 4–9
Example 4-8

For the transverse shear, using Eq. (4–24) with the correction factor \( C = 1.11 \) from Table 4–2, and noting that \( V \) is constant through the length of the beam,

\[
U_{\text{shear}} = \frac{CV^2l}{2AG} = \frac{1.11F^2l}{2AG}
\]

For the bending, since \( M \) is a function of \( x \), Eq. (4–23) gives

\[
U_{\text{bend}} = \int \frac{M^2}{2EI} \, dx = \frac{1}{2EI} \int_0^l (-Fx)^2 \, dx = \frac{F^2l^3}{6EI}
\]

The total strain energy is

\[
U = U_{\text{bend}} + U_{\text{shear}} = \frac{F^2l^3}{6EI} + \frac{1.11F^2l}{2AG}
\]

Note, except for very short beams, the shear term (of order \( l \)) is typically small compared to the bending term (of order \( l^3 \)). This will be demonstrated in the next example.
Castigliano’s Theorem

- When forces act on elastic systems subject to small displacements, the displacement corresponding to any force, in the direction of the force, is equal to the partial derivative of the total strain energy with respect to that force.

\[ \delta_i = \frac{\partial U}{\partial F_i} \quad (4-26) \]

- For rotational displacement, in radians,

\[ \theta_i = \frac{\partial U}{\partial M_i} \quad (4-27) \]
Example 4-9

The cantilever of Ex. 4–8 is a carbon steel bar 250 mm long with a 25-mm diameter and is loaded by a force $F = 400$ N.

(a) Find the maximum deflection using Castigliano’s theorem, including that due to shear.

(b) What error is introduced if shear is neglected?

Fig. 4–9
Example 4-9

(a) From Ex. 4–8, the total energy of the beam is

\[ U = \frac{F^2 l^3}{6EI} + \frac{1.11F^2 l}{2AG} \quad (1) \]

Then, according to Castigliano’s theorem, the deflection of the end is

\[ y_{\text{max}} = \frac{\partial U}{\partial F} = \frac{Fl^3}{3EI} + \frac{1.11Fl}{AG} \quad (2) \]

We also find that

\[ I = \frac{\pi d^4}{64} = \frac{\pi (25)^4}{64} = 19,175 \text{ mm}^4 \]

\[ A = \frac{\pi d^2}{4} = \frac{\pi (25)^2}{4} = 491 \text{ mm}^2 \]

Substituting these values, together with \( F = 400 \text{ N}, \ l = 0.25 \text{ m}, \ E = 209 \text{ GPa}, \) and \( G = 79 \text{ GPa}, \) in Eq. (2) gives

\[ y_{\text{max}} = 0.52 + 0.003 = 0.523 \text{ mm} \]

Note that the result is positive because it is in the same direction as the force \( F. \)

(b) The error in neglecting shear for this problem is \( (0.523 - 0.52)/0.523 = 0.0057 = 0.57 \text{ percent}. \)
Utilizing a Fictitious Force

- Castigliano’s method can be used to find a deflection at a point even if there is no force applied at that point.
- Apply a fictitious force $Q$ at the point, and in the direction, of the desired deflection.
- Set up the equation for total strain energy including the energy due to $Q$.
- Take the derivative of the total strain energy with respect to $Q$.
- Once the derivative is taken, $Q$ is no longer needed and can be set to zero.

\[
\delta = \left. \frac{\partial U}{\partial Q} \right|_{Q=0}
\]  
(4–28)
Finding Deflection Without Finding Energy

- For cases requiring integration of strain energy equations, it is more efficient to obtain the deflection directly without explicitly finding the strain energy.
- The partial derivative is moved inside the integral.
- For example, for bending,

\[
\delta_i = \frac{\partial}{\partial F_i} \left( \int \frac{M^2}{2EI} \, dx \right) = \int \frac{\partial}{\partial F_i} \left( \frac{M^2}{2EI} \right) \, dx = \int \frac{2M}{2EI} \frac{\partial M}{\partial F_i} \, dx
\]

- Derivative can be taken before integration, simplifying the math.
- Especially helpful with fictitious force \( Q \), since it can be set to zero after the derivative is taken.
Common Deflection Equations

\[ \delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE} \left( F \frac{\partial F}{\partial F_i} \right) dx \]  \hspace{1cm} \text{tension and compression}  \tag{4-29}

\[ \theta_i = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ} \left( T \frac{\partial T}{\partial M_i} \right) dx \]  \hspace{1cm} \text{torsion}  \tag{4-30}

\[ \delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI} \left( M \frac{\partial M}{\partial F_i} \right) dx \]  \hspace{1cm} \text{bending}  \tag{4-31}
Example 4-10

Using Castigliano’s method, determine the deflections of points A and B due to the force $F$ applied at the end of the step shaft shown in Fig. 4–10. The second area moments for sections $AB$ and $BC$ are $I_1$ and $2I_1$, respectively.

Fig. 4–10
Example 4-10

To avoid the need to determine the ground reaction forces, define the origin of \( x \) at the left end of the beam as shown. For \( 0 \leq x \leq l \), the bending moment is

\[
M = -Fx
\]  

(1)

Since \( F \) is at \( A \) and in the direction of the desired deflection, the deflection at \( A \) from Eq. (4-31) is

\[
\delta_A = \frac{\partial U}{\partial F} = \int_0^l \frac{1}{EI} \left( M \frac{\partial M}{\partial F} \right) \, dx
\]  

(2)

Substituting Eq. (1) into Eq. (2), noting that \( I = I_1 \) for \( 0 \leq x \leq l/2 \), and \( I = 2I_1 \) for \( l/2 \leq x \leq l \), we get

\[
\delta_A = \frac{1}{E} \left[ \int_0^{l/2} \frac{1}{I_1} (-Fx)(-x) \, dx + \int_{l/2}^l \frac{1}{2I_1} (-Fx)(-x) \, dx \right]
\]

\[
= \frac{1}{E} \left[ \frac{Fl^3}{24I_1} + \frac{7Fl^3}{48I_1} \right] = \frac{3}{16} \frac{Fl^3}{EI_1}
\]

which is positive, as it is in the direction of \( F \).
Example 4-10

For \( B \), a fictitious force \( Q \) is necessary at the point. Assuming \( Q \) acts down at \( B \), and \( x \) is as before, the moment equation is

\[
M = -Fx \quad 0 \leq x \leq l/2
\]

\[
M = -Fx - Q\left(x - \frac{l}{2}\right) \quad \frac{l}{2} \leq x \leq l
\]  \hspace{1cm} (3)

For Eq. (4–31), we need \( \partial M / \partial Q \). From Eq. (3),

\[
\frac{\partial M}{\partial Q} = 0 \quad 0 \leq x \leq l/2
\]

\[
\frac{\partial M}{\partial Q} = -\left(x - \frac{l}{2}\right) \quad \frac{l}{2} \leq x \leq l
\]  \hspace{1cm} (4)

\[\delta_i = \frac{1}{EI} \int \frac{1}{E} \left(\frac{\partial M}{\partial F_i}\right) dx \] bending  \hspace{1cm} (4-31)
Once the derivative is taken, $Q$ can be set to zero, so Eq. (4–31) becomes

$$
\delta_B = \left[ \int_0^l \frac{1}{EI} \left( M \frac{\partial M}{\partial Q} \right) \, dx \right]_{Q=0}
$$

$$
= \frac{1}{EI_1} \int_0^{l/2} (-Fx)(0) \, dx + \frac{1}{E(2I_1)} \int_{l/2}^l (-Fx) \left[ - \left( x - \frac{l}{2} \right) \right] \, dx
$$

Evaluating the last integral gives

$$
\delta_B = \frac{F}{2EI_1} \left( \frac{x^3}{3} - \frac{lx^2}{4} \right) \bigg|_{l/2}^l = \frac{5}{96} \frac{Fl^3}{EI_1}
$$

which again is positive, in the direction of $Q$. 
Example 4-11

For the wire form of diameter \( d \) shown in Fig. 4–11a, determine the deflection of point \( B \) in the direction of the applied force \( F \) (neglect the effect of transverse shear).

Fig. 4–11
Example 4-11

Figure 4-11b shows free body diagrams where the body has been broken in each section, and internal balancing forces and moments are shown. The sign convention for the force and moment variables is positive in the directions shown. With energy methods, sign conventions are arbitrary, so use a convenient one. In each section, the variable $x$ is defined with its origin as shown. The variable $x$ is used as a variable of integration for each section independently, so it is acceptable to reuse the same variable for each section. For completeness, the transverse shear forces are included, but the effects of transverse shear on the strain energy (and deflection) will be neglected.

Fig. 4–11
Example 4-11

Element $BC$ is in bending only so from Eq. (4–31),

\[
\frac{\partial U_{BC}}{\partial F} = \frac{1}{EI} \int_0^a (Fx)(x) \, dx = \frac{Fa^3}{3EI}
\]

(1)

Fig. 4–11
Example 4-11

Element \( CD \) is in bending and in torsion. The torsion is constant so Eq. (4–30) can be written as

\[
\frac{\partial U}{\partial F_i} = \left( T \frac{\partial T}{\partial F_i} \right) \frac{l}{GJ}
\]

where \( l \) is the length of the member. So for the torsion in member \( CD \), \( F_i = F \), \( T = Fa \), and \( l = b \). Thus,

\[
\left( \frac{\partial U_{CD}}{\partial F} \right)_{\text{torsion}} = (Fa)(a) \frac{b}{GJ} = \frac{Fa^2b}{GJ} \quad (2)
\]

For the bending in \( CD \),

\[
\left( \frac{\partial U_{CD}}{\partial F} \right)_{\text{bending}} = \frac{1}{EI} \int_{0}^{b} (Fx)(x) \, dx = \frac{Fb^3}{3EI} \quad (3)
\]

Fig. 4–11
Example 4-11

Member $DG$ is axially loaded and is bending in two planes. The axial loading is constant, so Eq. (4–29) can be written as

$$\frac{\partial U}{\partial F_i} = \left( F \frac{\partial F}{\partial F_i} \right) \frac{l}{AE}$$

where $l$ is the length of the member. Thus, for the axial loading of $DG$, $F_i = F$, $l = c$, and

$$\left( \frac{\partial U_{DG}}{\partial F} \right)_{\text{axial}} = \frac{Fc}{AE} \quad (4)$$

Fig. 4–11
Example 4-11

The bending moments in each plane of \( DG \) are constant along the length, with \( M_{DG2} = Fb \) and \( M_{DG1} = Fa \). Considering each one separately in the form of Eq. (4–31) gives

\[
\left( \frac{\partial U_{DG}}{\partial F} \right)_{\text{bending}} = \frac{1}{EI} \int_0^c (Fb)(b) \, dx + \frac{1}{EI} \int_0^c (Fa)(a) \, dx
\]

\[
= \frac{Fc(a^2 + b^2)}{EI}
\]  

Fig. 4–11
Example 4-11

Adding Eqs. (1) to (5), noting that \( I = \pi d^4/64, \ J = 2I, \ A = \pi d^2/4, \) and \( G = E/[2(1 + \nu)] \), we find that the deflection of \( B \) in the direction of \( F \) is

\[
(\delta_B)_F = \frac{4F}{3\pi Ed^4} \left[ 16(a^3 + b^3) + 48c(a^2 + b^2) + 48(1 + \nu)a^2b + 3cd^2 \right]
\]

Now that we have completed the solution, see if you can physically account for each term in the result using an independent method such as superposition.

Fig. 4–11